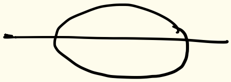
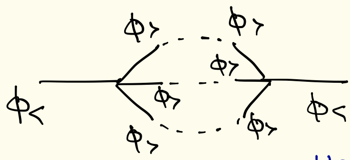
# Calculation of exponent $\eta$ at $O(\varepsilon^2)$

As discussed earlier, at $O(\varepsilon)$, the critical exponent $\eta = 0$. The first non-zero correction to $\eta$ comes from the diagram  which is $O(u^2)$ $= O(\varepsilon^2)$ at the critical point. lets calculate this.

The contribution to the action from this term is

$$\frac{u^2}{2} \int_0^{\Lambda/b} d^d k \; |\phi_<(k)|^2 \times N \times \int_{\Lambda/b}^{\Lambda} G_>(p)\, G_>(q)\, G_>(k-(p+q)) \; d^d p \; d^d q$$

where $G_>(p) = \dfrac{1}{t+p^2}$



one possible contraction

and $N = {}^4C_1 \times {}^4C_1 \times 3! = 96$

is the number of distinct diagrams that give the same contribution.

To extract exponent $\eta$ we note that a contribution $\dfrac{k^{2-\eta}}{2} |\phi(k)|^2$ for small $\eta$ may be written as $\dfrac{k^2 |\phi(k)|^2}{2} - \dfrac{\eta}{2} k^2 \log(k) |\phi(k)|^2.$

Therefore, we need to extract from the above diagram the coefficient of $\frac{k^2}{2} \log(k) |\phi(k)|^2$ term.

A trick to achieve this is to write integral in real space

$$\int d^d p \, d^d q \, G_>(p) \, G_>(q) \, G_>(k-(p+q))$$

$$\int d^d x \, G_>^3(x) \, e^{i \vec{k} \cdot \vec{x}}$$

Now $G_>(x) \sim \frac{1}{(2\pi)^2 x^2}$ in $d=4$ $\Rightarrow$ the integral is

$$\int \frac{d^d x}{(2\pi)^6} \frac{e^{i \vec{k} \cdot \vec{x}}}{x^6} \approx \frac{1}{(2\pi)^6} \int_{\Lambda^{-1}}^{k^{-1}} \frac{d^4 x}{x^6} \left[ 1 - \frac{|\vec{k}|^2 |\vec{x}|^2}{2 \times 4} + \ldots \right]$$

factor of 4 due to spherical symmetry.

The upper limit of the integral is set to $x \sim k^{-1}$ due to oscillating factor of $e^{ik \cdot x}$ while the lower limit is set to $\Lambda^{-1} \approx$ the lattice spacing. Now one may extract the coefficient of $k^2 \log(k)$ term as

$$\frac{1}{(2\pi)^6} \int_{\Lambda^{-1}}^{k^{-1}} \frac{d^4 x}{x^6} \frac{k^2 x^2}{8} \sim \frac{k^2 S_4}{8 (2\pi)^6} \int_{\Lambda^{-1}}^{k^{-1}} \frac{dx}{x}$$

$$\sim \frac{K_4}{8 (2\pi)^2} k^2 \log(\Lambda / k)$$

Where $K_4 = \frac{S_4}{(2\pi)^4}$

From this, we conclude that

$$\eta = \frac{96 u^{*2} K_4}{8 \times (2\pi)^2}$$

Using $u^* = \dfrac{\varepsilon}{36 K_4}$,

$$\eta = \frac{96\, \varepsilon^2}{8 \times (36)^2} \frac{1}{K_4 \times 4\pi^2}$$

$$K_4 = \frac{S_4}{(2\pi)^4} = \frac{2\pi^2}{(2\pi)^4} = \frac{1}{8\pi^2}$$

$$\Rightarrow \quad \eta = \frac{96\, \varepsilon^2}{4 \times (36)^2} = \frac{\varepsilon^2}{54}$$